NATIONAL CENTER FOR IMPROVING STUDENT LEARNING & ACHIEVEMENT IN MATHEMATICS & SCIENCE

inBrief

K-12 Mathematics & Science
RESEARCH & IMPLICATIONS
FOR POLICYMAKERS, EDUCATORS & RESEARCHERS SEEKING TO IMPROVE STUDENT LEARNING & ACHIEVEMENT

ABOVE: Elementary students work on solving mathematics problems and developing generalizations.

BUILDING A *Foundation* FOR LEARNING *Algebra* IN THE *Elementary Grades*

Teachers and researchers have long recognized that the transition from learning arithmetic to learning algebra is one of the major hurdles students face in learning mathematics. Researchers are finding, however, that elementary students can learn to think about arithmetic in ways that both enhance their early learning of arithmetic and provide a foundation for learning algebra. This *in Brief* highlights learning gains of 240 elementary students involved in a long-term study in Madison, Wisconsin,[1] and their remarkable ability to reason about arithmetic in ways that build their capacity for algebraic reasoning.

The study, led by researchers Thomas Carpenter and Linda Levi of the National Center for Improving Student Learning and Achievement in Mathematics and Science (NCISLA), found that innovative teacher professional development and refocused mathematics instruction paved the way for even first- and second-graders to begin to reason algebraically. Research results show that young students can learn to make and justify generalizations about the underlying structure and properties of arithmetic — generalizations that form the basis for much of algebra. Consistent with goals outlined in the National Council of Teachers of Mathematics (NCTM) *Principles and Standards* (2000), this NCISLA study shows that young children can begin building a foundation in algebra much earlier than typical curricula allow.

RESEARCH FOCUS: *The Early Algebra Research Project*

Currently in its third year, the early algebra project is led by NCISLA researchers Carpenter and Levi and 12 Madison Metropolitan School District teachers. Building on 15 years of Cognitively Guided Instruction (CGI) research,[2] Carpenter and Levi support teachers' professional development by systematically helping them to focus on students' mathematical thinking, in particular students' abilities to articulate, represent, and justify generalizations about the underlying structure and properties of arithmetic.

Building on student thinking. The researchers and teachers have found that students have a great deal of implicit knowledge about basic properties of arithmetic. For example, even first-grade students will choose to count on from the larger number to find a sum like $3+9$: These students implicitly recognize they can interchange the order from $3 + 9$ to $9 + 3$ to make the calculation easier. When these basic properties are not made explicit, however, many students are unsure whether the properties apply in new or unfamiliar problem contexts. For example, students may not understand that they can similarly change the order of the numbers if they are adding very large numbers, or fractions, or expressions involving variables in algebra. Other students may overgeneralize: They may, for example, interchange the order of numbers when subtracting or dividing.

Algebra builds on the same fundamental properties that form the basis for arithmetic. The abstract nature of algebra makes it even more

[1] *The research in Madison is providing a foundation for teacher professional development in schools in Phoenix, Los Angeles, and San Diego.*

[2] *The Cognitively Guided Instruction Professional Development Program engages teachers in learning about the development of children's mathematical thinking in particular content areas, building their own content knowledge, and refining instructional practice. For more information about CGI, see* in Brief *reference:* Children's Mathematics: Cognitively Guided Instruction *(a book and two multimedia CDs), 1999.*

important that students understand precisely when and why properties of arithmetic can be applied. The goal of the early algebra project is to help students make explicit and understand the underlying structure and properties of arithmetic — as they are learning arithmetic — so that they will have a solid base to build on as they go on to learn algebra with understanding.

Supporting professional development. A critical component of the early algebra project is teacher professional development. At summer workshops and meetings held throughout the year, the teachers and researchers analyzed the structure and basic properties of arithmetic and considered learning contexts that could encourage students to explicitly articulate generalizations about these properties. The group considered how students might think about specific problems and ways students might justify (or prove) their proposed generalizations were true. These sessions helped teachers better understand their students' thinking and build their own mathematics knowledge. In addition, the meetings supported the evolution of a committed and growing professional community.

Enhancing classroom instruction. True–false and open number sentences were the primary means of eliciting generalizations from students. (See "Number Sentences Used to Generate Generalizations".) Teachers used true–false sentences to initiate discussions that focused on *how* students knew a sentence was true or false. This method generally was sufficient to elicit generalizations from students in the class. The class then discussed these generalizations and whether they were always true for all numbers, which led to an extended classroom analysis of what is required to justify a generalization. This form of instruction built on student thinking and supported their understanding of basic properties of arithmetic required for algebra. The following classroom excerpts illustrate how these goals were accomplished.

NUMBER SENTENCES USED
to Generate Generalizations

Below are examples of number sentences that teachers used to help students articulate generalizations about zero and multiplication.

Examples: 78 + 0 = 78; 23 + 7 = 23*

"When you add zero to a number, you get the number you started with."

Examples: 96 – 96 = 0; 74 – □ = 74

"When you subtract a number from itself, you get zero."

Examples: 96 × 0 = 0; 43 × 0 = 43*

"When you multiply a number times zero, you get zero."

Examples: 65 × 54 = 54 × 65; 94 × 71 = 71 × □

"When multiplying two numbers, you can change the order of the numbers."

* denotes a false number sentence

First- and Second-Graders' Capacity for Algebraic Reasoning

Students from a first- and second-grade class were given a set of problems to guide them into expressing a generalization about what happens when zero is added to a number. The children were not only able to articulate the generalization; they were able to take the discussion to a higher level mathematically.

The teacher asked the students if "78 – 49 = 78" was true or false. The students immediately responded:

CHILDREN: False! No, no false! No way!

TEACHER: Why is that false?

JENNY: Because it is the same number as in the beginning, and you already took away some, so it would have to be lower than the number you started with.

MIKE: Unless it was 78 – 0 = 78. That would be right.

TEACHER: Is that true? Why is that true? We took something away.

STEVE: But that something is, there is, like, nothing. Zero is nothing.

TEACHER: Is that always going to work?

LYNN: If you want to start with a number and end with a number, and you do a number sentence, you should always put a zero. Since you wrote 78 – 49 = 78, you have to change a 49 to a zero to equal 78, because if you want the same answer as the first number and the last number, you have to make a zero in between.

TEACHER: So do you think that will always work with zero?

MIKE: Oh, no. Unless you 78 minus, umm, 49, plus something.

ELLEN: Plus 49.

MIKE: Yeah. 49. 78 – 49 + 49 = 78.

TEACHER: Wow. Do you all think that is true? [All but one child answered yes.]

JENNY: I do, because you took the 49 away, and it's just like getting it back.

[Emphasis added to lift out teacher's question strategy.]

As the discussion above continued, the group collectively came up with the generalization: "Zero added to another number equals that other number." They also came up with the generalizations: "Zero subtracted from another number equals that number," and "Any number minus the same number equals zero." The students not only applied these generalizations to solve problems involving zeros, they also came up with number sentences that embodied more complex ideas (e.g., 78 – 49 + 49 = 78).[3]

The study shows that even first- and second-grade children are able to argue about mathematical concepts and operations in ways not generally expected of students at this age. These students were able to express generalizations and reason about them.

Third- Through Fifth-Graders' Introduction to Mathematical Proof

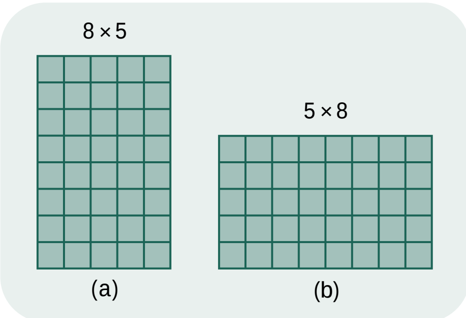
Third- through fifth-grade students participating in the early algebra project not only identified more complex generalizations, they also were challenged to *justify* their generalizations using arguments that helped them to gain an appreciation for mathematical proof. They learned that justification went beyond proposing an endless supply of examples for which the generalization applied.

For example, Mary Bostrom's third- and fourth-grade students were asked to justify the generalization: "When you multiply two numbers, you can change the order of the numbers" ($a \times t = t \times a$). The students initially calculated a lot of examples, such as 8 × 5 = 5 × 8, but Ms. Bostrom pressed them to show that the generalization was true for all numbers, not just some.

To prove the generalization was always true, one pair of students went back to the basic definition of multiplication using linking cubes to illustrate a specific example (8 × 5 = 5 × 8, see Figure 1). After some discussion, the students provided a concrete justification, demonstrat-

[3] *Students at first used everyday language to express generalizations. After students were introduced to variables and open number sentences, they were able to express generalizations in open number sentences that were always true, such as* $b - b = 0$, *and* $b - a + a = b$.

FIGURE 1. Proving 8 × 5 = 5 × 8: Two students' proof using the rotation of cube arrays.



ing that they could rotate one group of cubes (Fig. 1a) 90 degrees and place the rotated array over the other group (Fig. 1b). "Look, they are the same," exclaimed one student. The other student added, "That works, but you don't even have to have the other one [Fig. 1b]. You can just turn this one [Fig. 1a] and see the other groups."

Although the students used a specific example to justify the generalization, the way they explained the example showed that they understood that they could do the same thing with any numbers. These students were beginning to learn what was required to justify that a generalization was true for all numbers.

Students' Understanding of Equality

An understanding of equality and the appropriate use of the equal sign is critical for expressing generalizations and for developing algebraic reasoning. For example, the concept of equality (indicating a relationship between different parts of an equation and meaning "the same as") is embedded in a number sentence like $8 \times 5 = 5 \times 8$. Unfortunately, many students typically hold misconceptions about the meaning of the equal sign (see Kieran, 1981; Matz, 1982) and tend to think it means merely to carry out an operation.

In order to assess students' understanding of the meaning of the equal sign, the researchers and teachers gave the students a seemingly simple number sentence:

$$8 + 4 = \square + 5$$

Consistent with previous research (Kieran, 1981; Saenz-Ludlow & Walgamuth, 1998), most students responded that either 12 or 17, rather than 7, should fill in the box. They thought either that the number immediately after the equal sign had to be the answer to the calculation or that they should just add all the numbers together. Furthermore, teachers participating in the research found that explaining the equal sign was not sufficient to ensure that students understood its meaning. Building off students' different conceptions of what the equal sign meant, teachers engaged students in mathematical discussions that helped them confront their misunderstandings and achieve an accurate understanding of the meaning of the equal sign.

NUMBER SENTENCES THAT
Challenge Students' Conception of Equality

a) $7 = 3 + 4$

b) $8 = 8$

c) $5 + 8 = 8 + 5$

d) $8 = 5 + 13$ *

* denotes a false number sentence

Across several class periods, the teachers continued to provide examples of true and false number sentences that challenged students' conceptions of the meaning of the equal sign and reinforced their learning. These discussions resulted in significant changes in student understanding and problem-solving improvements (see Figure 2).

IMPLICATIONS: *Reform of Elementary Mathematics Instruction and Teacher Professional Development*

This study showed that young students can learn arithmetic in ways that provide a foundation for learning algebra. Recognizing young students' ability to reason algebraically does not suggest that elementary students should learn high school algebra. Rather, this study showed that a broader conception of algebra can be a part of elementary instruction that builds on students' implicit mathematical knowledge and increases their ability to understand, reason, and engage in challenging problem solving.

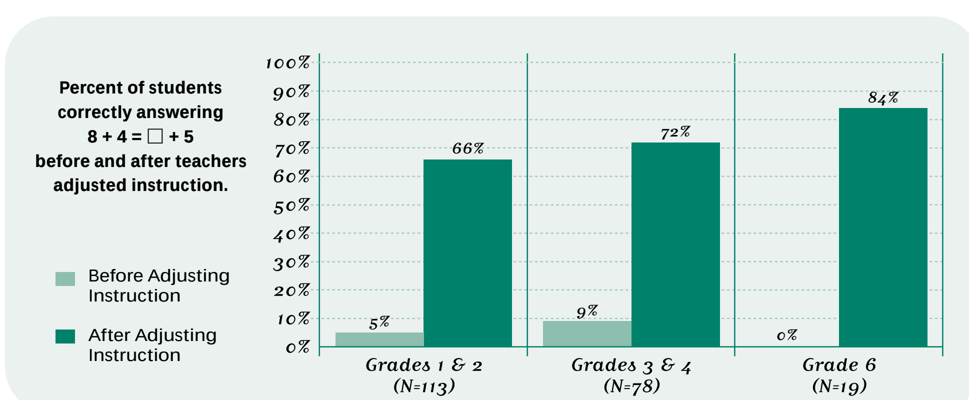
The instructional strategies outlined here have significant implications for teacher learning and professional development. Through their ongoing work with fellow teachers and researchers, the teachers gained essential insight into their students' mathematical reasoning; they also forged a community through which they gained mathematical knowledge and crafted problems that benefited their students' learning. Although it requires a significant commitment and support from schools and policymakers (see insert on Policy Considerations), participants in both CGI and the early algebra project have seen this form of professional development yield exciting results in students' learning of mathematics. Interested educators should also refer to the Teaching Considerations insert, which specifies some resources and strategies teachers might adopt with their students and colleagues in their educational community.

For More Information

A research report about the early algebra project and other relevant publications are listed in the reference section of this *in Brief* and are available at the NCISLA website at http://www.wcer.wisc.edu/ncisla/.

Researchers Tom Carpenter and Linda Levi can be reached through the National Center for Improving Student Learning and Achievement in Mathematics and Science at the University of Wisconsin–Madison, 1025 W. Johnson St., Madison, WI 53706; (608) 263-3605; E-mail: ncisla@education.wisc.edu

FIGURE 2. Students' Increase in Understanding of the Meaning of the Equal Sign



NATIONAL CENTER FOR IMPROVING STUDENT LEARNING & ACHIEVEMENT IN MATHEMATICS & SCIENCE

*in*Brief

K-12 Mathematics & Science
POLICY CONSIDERATIONS
Elementary Students Can Reason Algebraically

The research summarized in this *in Brief* shows that with appropriate teacher professional development, teachers can learn to help students learn mathematics in ways that both enhance their understanding of arithmetic and provide a foundation for learning algebra. Policymakers and administrators can support the development of this kind of instruction in their schools by —

- Supporting sustained, long-term programs of professional development that focus on the development of students' mathematical thinking, as well as teachers' understanding of the related mathematical ideas.
- Supporting the development of professional communities that help teachers make their classrooms places for both students and teachers to engage in inquiry and learn with understanding.
- Encouraging teachers to conceive of teaching in terms of understanding and developing students' mathematical thinking.

Supporting Sustained, Long-Term Professional Development

There are no simple solutions for the problems facing our schools. Simply introducing a new curriculum or providing teachers a one-week workshop is not going to result in sustainable improvement in teaching or significant gains in student achievement. Fundamental change requires that teachers develop a deep understanding of the mathematics they teach and the ways their students think about and learn that mathematics. Developing that understanding takes time and requires sustained, long-term professional development. School and district leaders need to find ways to provide resources to support those kinds of long-term professional development opportunities.

One of the most critical resources is time. Teachers need time to participate in the professional development, time to meet with other teachers to discuss what they are learning and how it is reflected in their classroom interactions with students, as well as time to reflect on their teaching and their students' understandings and misconceptions. Teachers also need access to resources, including people and materials, that can help them develop their understanding.

ABOVE: Elementary students reason together about mathematics problems.

School and district leaders can also support professional development by participating in it themselves. Leaders need to understand the ideas that the teachers are learning so that they can understand what is going on in the teachers' classrooms and provide the necessary support and assistance when called on. Furthermore, the participation of all interested parties communicates to the teachers that their professional development is important and valued by the school and district.

Supporting Professional Communities

Teachers participating in the NCISLA early algebra project entered into the project at varying levels of knowledge and mathematics confidence; they also came into it from different types of schools and classrooms (in terms of grade level and student diversity). Together with the researchers, these teachers committed to forming a professional development community. They met every month and focused their discussions on students' ways of solving group-selected mathematics problems and number sentences. The teachers themselves were "learners" — learning about students' thinking and learning mathematics. School and district leaders can foster the development of these kinds of communities by making time available for teachers to meet, by arranging opportunities for teachers to have common planning time as they work to develop such a community, and by being active participants in the communities.

Focusing on Students' Mathematical Thinking — Shared Vision

School and district leaders can help teachers focus on developing students' mathematical thinking by making that focus a priority in the school. By talking to students about their mathematics and by talking to teachers about their students' mathematical thinking, principals can help create an atmosphere in which such conversations become the norm. One principal in the project asked teachers to regularly bring examples of their students' work to her and discuss with her what the students understood and were learning. Teachers' analysis of their students' thinking was included as a significant part of the teachers' annual review. Questions about candidates' understanding of and interest in understanding students' mathematical thinking might also be an important component of hiring interviews.

For More Information

Policymakers who would like more information about student learning and sustainable teacher professional development can contact the National Center for Improving Student Learning and Achievement in Mathematics and Science (NCISLA) at (608) 263-3605 or refer to the NCISLA web site: www.wcer.wisc.edu/ncisla. Researchers Tom Carpenter and Linda Levi can also be reached at the NCISLA.

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inBrief

K-12 Mathematics & Science TEACHING CONSIDERATIONS

Elementary Students Can Reason Algebraically

The research summarized in this *in Brief* showed that elementary students can learn to think about arithmetic in ways that both enhance their learning of arithmetic and provide a foundation for learning algebra. If you want to apply these ideas in your classroom, consider the following:

Build a Foundation for Learning Algebra

- **Ask questions** that provide a window into children's understanding of important mathematical ideas. For example, students' responses to the number sentence "9 + 6 = □ + 8" tells a great deal about their understanding of the meaning of the equal sign. Probe students' reasons for their answers. Ask students *why* they answered as they did.
- **Provide students opportunity to discuss** and resolve different conceptions of mathematical ideas. For example, the different conceptions of the equal sign that emerge from students' solutions to the open number sentence "9 + 6 = □ + 8" can provide the basis for a productive discussion.
- **Provide students with equations** that help them understand that the equal sign represents a relation between numbers, not a signal to carry out the preceding calculation. Examples include "□ = 8 + 9," "8 + 6 = 6 + □," "9+6=□+8." Vary the format of number sentences: Include sentences in which the answer does not come right after the equal sign.
- **Provide students with true and false number sentences** that challenge their misconceptions about the equal sign (e.g., 8 = 5 + 3, 9 = 9, 7 – 4 = 7 – 4).
- **Provide students problems that encourage them to make generalizations** about basic number properties (see "Number Sentences to Elicit Generalizations".) When they provide an answer to one of the problems, ask them how they know their answer is correct. That often will result in their stating a generalization such as "When you subtract a number from itself, you get zero." When they do state a generalization like this, ask, for example, "Is that true for all numbers?"
- **Have students justify generalizations** they or their peers propose (see page 3 for an example). Justification of generalizations requires more than providing a lot of examples (e.g., 8 × 5 = 5 × 8). By expecting children to justify their claims, you can help them gain skills in presenting mathematical arguments and proofs. Use the questions "Will that be true for all numbers?" and "How do you know that is true for all numbers?" repeatedly to encourage students to recognize that they need to justify their claims in mathematics.

NUMBER SENTENCES
to Elicit Generalizations

Is this number sentence true or false?

97 – 97 = 0	48 × 0 = 48 *
37 + 58 = 58 + 37	56 ÷ 0 = 0 *

What number can you put in the box to make this a true sentence?

□ + 74 = 74	35 × □ = 8 × 35

* denotes a false number sentence

Form a Teaching Community Focused on Students' Mathematical Thinking

- **Make classrooms a place where both you and your students are learning.** Engaging in regular inquiry about students' mathematical thinking can be one of the most powerful forms of professional development. Such inquiry does not, however, thrive in isolation. Seek out other teachers who share your interest in talking about students' thinking and share with one another the interesting observations you are making about students' mathematical thinking in your classes.
- **Form a community with teachers and other resource people.** You will find the support invaluable. Together with the researchers, the teachers participating in the NCISLA early algebra project committed to forming a professional development community. They met every month and focused their discussions on students' mathematical thinking, ways to elicit that thinking, how to figure out what students understood, and ways to engage students in discussing mathematical ideas. The teachers themselves were learners — learning about students' thinking and learning about mathematics.

For More Information

Elementary teacher resources and publications are available at www.wcer.wisc.edu/ncisla. See:

Teacher learning and professional development: The NCISLA fall 1998 newsletter *Principled Practice: Teachers as Learners*, at www.wcer.wisc.edu/ncisla.

Children's understanding of equality: *Children's Understanding of Equality: A Foundation for Algebra*, by Karen P. Falkner, Linda Levi and Thomas P. Carpenter, in *Teaching Children Mathematics*, vol. 6, no. 4, Dec. 1999. (Also at www.wcer.wisc.edu/ncisla/teachers)

Cognitively Guided Instruction: *Children's Mathematics – Cognitively Guided Instruction* (with two multimedia CDs), by Thomas P. Carpenter, Elizabeth Fennema, Megan Loef Franke, Linda Levi, and Susan B. Empson. Heinemann Publishers, 1999.

ABOVE: An elementary student explains his reasoning to his teacher and fellow students.

ABOUT *in Brief*

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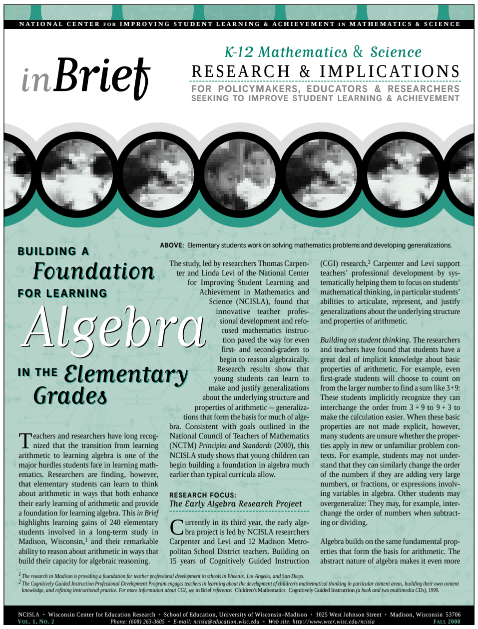

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