Linear Measurement:
Designing and Redesigning Instruction Based on Formative Assessment

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This chapter describes a yearlong classroom action research project on linear measurement with a class of 25 fifth grade, economically underprivileged students. My goal was to help these students make sense of the mathematics of linear measurement, partly because measurement is featured in statewide and national standards and partly because I thought that measurement would provide good foundations for related forms of mathematics. I used formative assessment (ongoing assessment embedded in the texture of my instruction) to make instructional decisions and to redesign the curriculum. Much to my surprise, helping students develop understanding of measurement was a difficult enterprise, fraught with nuances and layers of conceptual development that I did not anticipate. In the process, I re-considered my understanding of measurement too.

The tasks that I used as windows to student thinking were designed as formative assessments that made the core ideas of measurement "problematic." Students can use a ruler proficiently without ever stopping to consider the nature of the units that they are counting, or even the conceptual underpinnings of iteration (translation of units) that allow counts to represent measures. So, I invented or borrowed problems and situations that I hoped would entice students to grapple with fundamental principles of measurement—to make something previously experienced as transparent, problematic. Sometimes I posed solutions invented by one or more students for whole-class consideration. The debate and discussion that followed helped students to examine their own ideas and change, integrate, modify and/or solidify their thinking.

Although I continually focused on student thinking, the formative assessments evolved considerably. To bootstrap the design process, I took copious notes of student reasoning and problem solving strategies. I conducted case studies of several studies representing a range of mathematical skill. I continually asked what I knew about children's understanding and, when possible, consulted with other teachers and researchers in the project to get their impressions of student work. I then revised or invented new task-assessments to build on what I thought students knew, so that I could pose new challenges for them to consider. As I worked, I discovered that I did not fully understand the complexity embedded in the “big ideas” of linear measure, and at times, how to make sense of the students’ thinking enough to help them take fruitful next steps. A number of times, I misinterpreted what I saw, causing both students and me to struggle in ways that may not have been necessary. I even might have created obstacles to understanding. However, I believe that my experiences are likely not unique, so I’ve decided to chronicle my journey.

Initial Impressions of Student Understanding

Before I began work on linear measurement, students tried a few items assessing their understanding of linear measure (see Figure 1). I expected that students would do reasonably well. Instead, not one student responded correctly to any of these items! When I followed up with individual students, thinking that there must be something wrong with the items, students’ responses revealed that they did not understand partitions of units other than one-half, and they did not understand that any point on the ruler can serve as the origin (zero). For nearly all, measurement was counting a number of markings, usually only in whole numbers. For example, most of the students thought that the length in B was 3 or 4 units long.

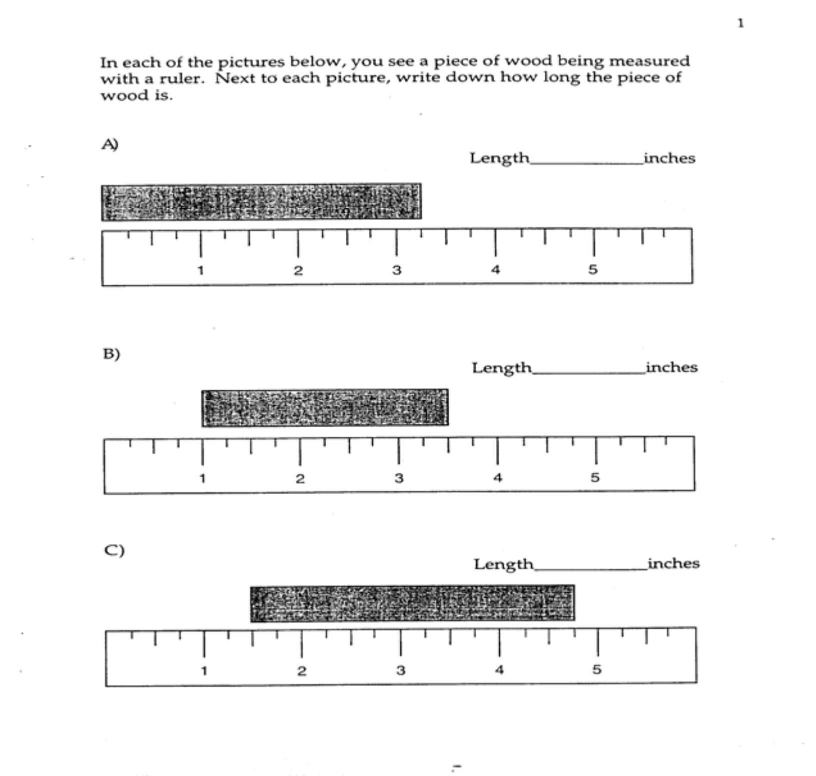


Figure 1. Pre-Test Items

I was surprised by the comparative paucity of children's understandings, partly because in previous grades, students practiced measuring lengths with rulers. Apparently, very little had "stuck." To tackle this problem, I decided to find out more about how students were thinking about the nature of units of length measure. I focused on:

- Units – The nature of units, including the importance of congruent (identical) units, the role of standards, and the process of iteration (units can be reused).
- Partitioning - Units can be subdivided or partitioned,
- Composition- "Units of units" can be composed, so that, for example, 6 inches can be read as 3 2-inch units.
- Zero point – The point of origin of a ruler is zero. The interval is invariant when translated, so that length measure is a distance traveled.

Getting Started

I posed a series of tasks: pacing, foot-tapes, and personal-unit tapes. Each was designed as an assessment—to help me better understand how students were thinking—and as instruction—as tools to help students begin to make sense of big ideas. Each of these tasks played a different role and revealed or developed different aspects of the students' understanding. And, they occasioned a few "side excursions" into symbolization and fractions.

Pacing

I decided to begin with an approach to measurement that would anchor linear measure to children's bodily experiences of walking. Walking draws on conceptions of measure as motion and travel, and so seemed especially well suited as a starting point. Students worked in groups to determine the distance between different places on the playground by pacing, using their feet as the measuring tool. Their efforts to measure with their feet shed some light on their conceptions of measure.

Conceptions of units. Units were literally enacted. Most students stepped with their feet end-to-end, leaving no gaps. Most also were immediately concerned that using different foot sizes to measure was problematic: "If you want to measure a distance, the units must be equal in size. A student with a small foot would not end up with the same measure as a student with a larger foot." One group decided to use one student's

(Richard's) normal pace to measure the distances. During the class discussion, the students were concerned that Richard was leaving spaces between his footsteps. Richard explained that he had worked hard to keep his footsteps consistent distances apart, but the class wasn't convinced that he could do that. They suggested that therefore his measurements were not accurate. This suggested to me that students understood that units of measure should be identical and that units ought to fill the space (or if they don't, the gaps ought to be consistent). This immediately suggested some conceptual resources that I could capitalize on in the future.

Conceptions of zero point. However, much to my surprise, many of these fifth-grade students did not know where to start counting when they paced: "Was it when a step is taken or does the first step (where their feet are) count? "If you start with your foot in front of the line, is that a step? Or do you have to start behind the line so that the first step taken is in front of the line?" Comments like these indicated that students were confused about the origin of the measure. Moreover, they counted units as the action of stepping, without seeming to think of the units as being distances traversed between the points on the playground being measured (see Figure 2).

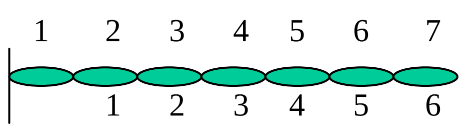


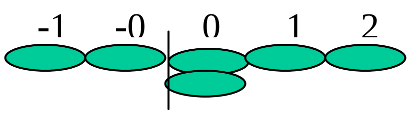
Figure 2. Competing ideas about counting foot units.

When the class began to grapple with this critical concept, I did not take full advantage of the opportunity before us and moved on before they could solidify or mathematize their thinking, even at this basic level. At the time, I was satisfied with the idea that all steps had to be counted. I believed that if I could focus on steps taken, then other concepts would fall into place. Little did I know that the problem posed in Figure 2 was a window into challenges the class would later revisit in different guises.

*Conceptions of partitioning units***.** Playground pacing also revealed a reluctance of the students to deal with the left over spaces. I asked the students how they dealt with the ending point when they measured. The class, for most part, was not at all concerned with any space that was less than a foot. The few who were concerned called any leftover distance one half. Of course, in retrospect I realized that I did little to make greater precision necessary, either in the design of the task or in the subsequent conversation.

Walking on the Number Line

The difficulty with representing paces revealed by the previous activity gave me pause, so I decided to explore further how students might represent pacing. Rich Lehrer and I decided to ask students to symbolize different kinds of walks. Rich took one step and asked the class: What that was? *"One step."* Then Rich took 4 steps forward and asked what that would be and how might it be symbolized: "Five because 4 + 1 = 5." I wrote this number sentence on the board. Standing at five, Rich asked where would he would be if be took 3 steps back. Somewhat to our surprise, several suggested a negative sign to show the change in direction, getting up and writing 5-3=2. I introduced a whole number-line representation, with associated operations of + representing forward movement and - representing backward movement, with zero as the origin. We took a few walks and represented them, apparently unproblematically, on the numberline. Having established a correspondence between direction and sign, Rich decided to try to see what students would do if he suggested moving one step back from zero. To my surprise, this ignited a storm of controversy.



Jamie proposed that the first step back would be a -0 and the second, a -1. A step forward would be 1 and a second step forward would be 2. I responded to Jamie's diagram by asking:

T: Can you draw the same thing without feet?

Mario: -1 -0 0 1 2

T: What is the difference between 0 and -0?

Mario: Zero is me standing in the middle and -0 is stepping back one

T: I am confused - how come the picture is not showing that?

Richard: I disagree. Try -2 -1 1 2 I took a step backward and that was something, not nothing. Then he modified his symbolization to -3 -2 -1 0 1 2 3

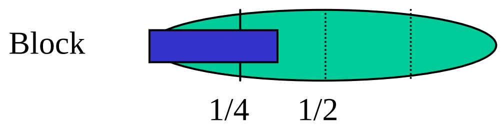
At this point, the class reached an impasse, some using an unsigned zero as the origin and a -0 as the first step, while others had decided that Richard's way (designating zero as origin and -1 as a step back from that origin. At this point, Jamie excitedly pointed to an integer (including negative numbers) number line on the far side of the room: "Look how they do it!" I asked: "Why do you think they do it that way? Jamie ventriloquated Richard: "You step backwards and that's -1, not nothing." This episode alerted me to potential links between geometry, measurement, and number, with measure serving as the mediator between the geometry of the line and counting numbers. It also alerted me to ongoing opportunities to symbolize measurement, a process which was not as straightforward as I had anticipated.

Making a Foot-Tape

I followed up on pacing by asking students to construct a foot-strip ruler using their traced footprints. I hoped to "lift away" from the plane of activity to have students explicitly consider the nature and role of units in linear measure. I wondered how students might symbolize what they had previously understood in activity.

*Conceptions of units***.** The students recalled that different people had counted different number of footsteps for the same distances, so they anticipated that perhaps this could be resolved by using one person's foot as the standard. For some, their markings on the foot-tape corresponded to cut-outs of this foot. When they ran out of tape, they marked the spot with their finger and continued to iterate footstep units. However, others confounded me. Rosaura, who argued strongly in the first task that the feet had to be the same size to get accurate measures, surprised me when she did not use that thinking to create partitions on her foot-print measuring tool. She did not connect understandings verbalized during the activity of pacing to their counterparts in a system of representation. Instead, she drew lines arbitrarily across her footprint tool to mimic the lines on a ruler. It was as if she saw no relationship between the act of measuring and the creation of a ruler. Rosaura's case was typical of many students, whose grasp of measurement in action was not paralleled by their grasp in representation of their actions.

Conceptions of partitions. When measuring with the foot-strip ruler, some students folded their rulers repeatedly to make equal partitions so they could measure items or distance smaller than a foot. Very few students understood what to call these partitions. Most of the class could identify 1/2 of a *foot* but beyond that could not use the partitions to describe the measure. Only two students were able to deal with fractions other than 1/2. Jamie created sixteenths by repeated halving and was able to describe objects measured using equivalent fractions, which was more sophisticated than I anticipated. *"The board was 9 4/16 or 9 1/4 footprint measuring tools long."* Another student, Cesar, folded his footstrip in half and then half again. When measuring a small wooden block, he described it as being 1/4 and a half of a fourth. Although this was a sophisticated response Cesar did not have enough experience with fractions to find out what 1/2 of 1/4 would be.



Personal Units

I decided that I wanted to revisit students' conceptions of unit in a new context, to see what they might have made of our previous conversations and lessons about pacing and building foot-strip rulers. Small groups were provided a set of units ranging from two to six inches long. Students named the units, often after classmates, something that I borrowed from Carmen Curtis, a teacher in Verona, WI. Students measured objects and distances in the room using these nonstandard units, with the goal of creating "very accurate" measures. The distances varied from the length of a paper clip to the width of the room. I was especially interested to see how students dealt with running out of units (creating the need for iteration) and how they measured objects where the length was not a whole number quantity (creating the need for partitioning units). Moreover, I knew that measuring large objects or distances would be problematic for students with the smallest personal units, so that composite units might be more efficient than the original units of measure.

Conceptions of iteration. Most groups of students reused units when they ran out, but in one group, an issue of order of reuse arose. The group created a "train" of units to measure objects and distances. When the units ran out, they would take the one from the back of the train, move it forward and continue the count. At one point, one student pulled a unit from the middle of the train and moved it forward. The group did not agree, and one student explained that you could not reuse the units in random order. When he was challenged by other members of the group, and asked to try it both ways to see if the measurement would be the same, he tested the idea and found that both ways yielded the same measurement. Although this was convincing to him, he maintained that more was at stake—that there must be a consistent method of measure.

Composite units. In some groups, students responded to pragmatic problems of measuring long distances with small units by constructing composite units. They took a ruler, yardstick, or a piece of paper and figured out how many of their units fit on that object. Then they measured the distance using their composite unit and calculated the quantity in the original unit.

Conceptions of zero point. One group had difficulty determining the length of a social studies book. This group taped their units together to make a measuring tape. They laid the measuring tape on top of the book, noticed that the left over space was about one-fourth of a unit, but could not decide how to quantify the length (See Figure 3). They appeared to confuse measuring with counting, so noticing that they were in the fifth count, their solution was to call the length five and one-fourth.

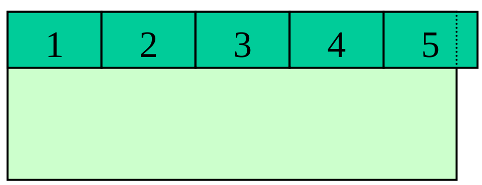


Figure 3. Zero-point confusion measuring the length of a book

This confusion exemplified difficulty reconciling distance traversed with the origin of the scale, a wonderful teachable moment that I did not pursue. Having the class grapple with this idea would have given the class the opportunity to begin to think about what zero point means. I did not pursue this because I did not fully understand the complexity of the idea: My thinking was anchored to clearer examples like moving the origin from 0 to 1, as it is represented in the second item displayed by Figure 1. I did not realize then that zero point issues would be revealed repeatedly at different levels of complexity and in different contexts.

Partitioning. Most of the measurement results of the data collection were reported in eighths and sixteenths, which were established by repeated halving of the paper strip units. This was a remarkable advance especially because half of the students in the previous activity had drawn in unequal partitions and the other half struggled to correctly label their partitions. Apparently experience with repeated splitting of the units provided grounding for establishing conceptions of part-whole relations.

Further Investigations

At this point, I was confident that the class was making some headway, but as I thought about each student rather than the "class," I was struck by how little I knew about individuals. So, I designed a series of follow up assessments aimed at helping me see how individual students might be thinking. One form of assessment was aimed at exploring how students were thinking about creating systems of representation. I had pretty good evidence that students could iterate units, and partition them, but I had some nagging concerns about how they might symbolize this activity. So, I asked them to create a ruler, given a unit and a strip of paper.

Much to my surprise, I found that many of the students who could arrange units and even partition them in the previous activity still had difficulty representing these qualities of unit in their designs of rulers. Richard's work was a prime example. He used the unit to draw equal units on his ruler, but then drew partitions of units freehand. His method created space at the end of the paper strip that could not be filled by a whole unit. He reverted to paper folding of the unit to measure the remaining space and labeled it 1/3, because he had 3 equal partitions on his folded paper. This suggested some understanding of the importance of congruent partitions, but this understanding was not yet linked to the overall design of a ruler.

I designed a second form of assessment to elicit students' ideas about zero-point. They measured the length of the model car displayed in Figure 4, with an origin at 2. All of the students began their measurement at the back end of the car, but about half of the students struggled with the zero point concept. In the example in figure 4, the student partitioned the last unit as two fourths, but relabeled the number on the line beneath the back end of the car as 1, not 0. When I followed up, he suggested that the 1 was referring to the first unit, so that the measure of the entire car should be five and two quarters of the unit. I did not expect this reasoning, and further questioning convinced me that he was thinking that labeling the starting unit as 1 was a good general strategy for compensating for not starting at "the beginning."

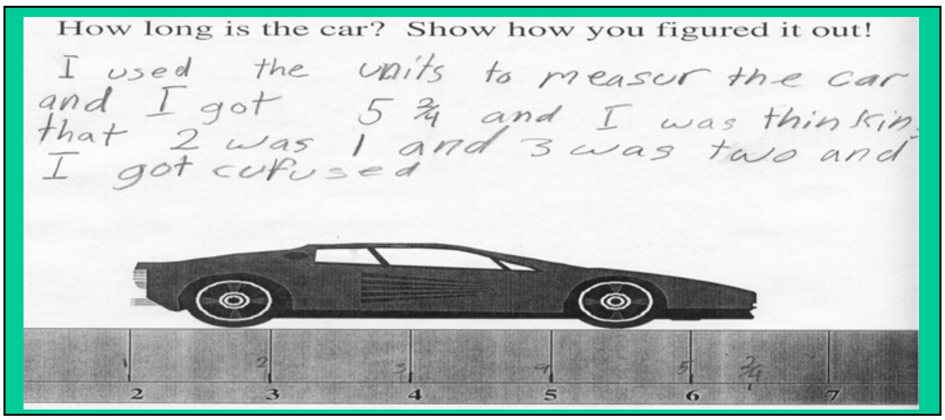


Figure 4. Assessing Zero Point

Creating Contrasting Cases for Eliciting Students' Thinking

After looking over the students' measurement assessment, I became concerned with the number of students who still had difficulty with representing linear measurement. Moreover, I wondered how students' might think about the origin of a scale in different contexts. Hence, I posed several "case studies" of ways of thinking for the class to consider. These were drawn from students' work that exemplified prototypical, but contrasting, ways of thinking about these issues.

Footprints and Zero Point

Students considered how they had labeled their footstep ruler. I wanted to invoke the context of pacing again to emphasize translation through a distance. Cesar shared two strategies that he used to label his ruler on a transparency (Figure 5). He distinguished between counting units and counting space, as follows:

Cesar: At first I was only counting the squares instead of the space. (He draws in the numbers on the bottom of the ruler. Then he labels the top as shown on the ruler above to show the end point of each unit.)

T: So what is the difference between the two ways that he labeled his ruler?

Jose: He points to the top line and says, here I think he did it wrong because he doesn't count the steps but on the bottom line he did it right because he is counting the steps. [Jose focuses on units as stand-ins for paces.)

T: Where are the steps, Jose?

Jose: This is a step (pointing to a box), this is a step...each of the boxes is a step.

T: So if your boxes are your steps, where is the end of the first step?

Daniel: (Goes up and points) The lines on the bottom show the steps or the boxes and the ones on top here show the end of the steps.

Maria: I think that the top and the bottom are the same.

T: Where is the starting point?

Jose: He points to the ruler and shows the back line on the ruler and points out the first box, one, then two, three (going across counting the boxes).

At this point, it was evident that some of the children, like Jose, were looking at a unit as a tangible object, something to be counted, not as a measure of a distance. I intervened to help children reconsider points of departure and arrival, rather than objects:

T: (Writes the numbers inside the box to see what the students thought about numbering a ruler or measuring that way.) That is what I was picturing (Figure 5) that Cesar was doing with the labeling on the bottom of the ruler. It is something that I have seen on a lot of different papers in this classroom, so what I am trying to find out is what is it that you are thinking about. Is it okay to have the number in the middle of the box, or does it need to go at the end where Cesar drew it on the top of the ruler? What is your thinking about that? Does it matter where you label your numbers?

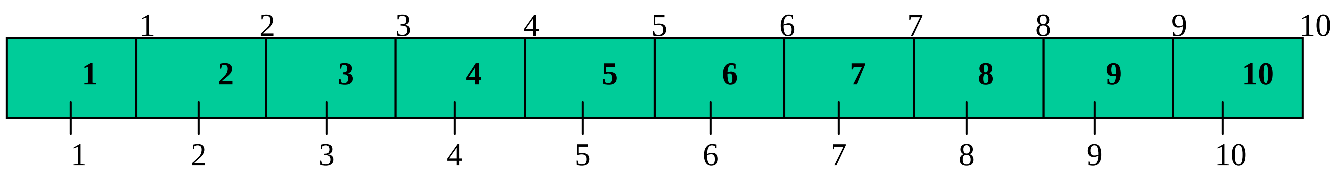


Figure 5. Cesar's Ruler

T: Where is the zero on this ruler?
Richard: I am thinking right here (pointing to the beginning edge of the ruler).
T: So this is the starting point of the ruler.... then, where does step one end?
Richard: Goes up and points to the top line of the ruler where Cesar labeled the end of the first unit.
T: Why? Where does the number one belong?
Richard: (points to the first box) Right here (other voices echo in the background).
T: It's in three different spots, which one is where it should go?
Richard: The one on top.

In many of the problems presented, students had counted boxes. Although I was not initially concerned about this, Jose, and students like him, helped me realize that many of the children in the class were counting objects, not thinking of movement across a distance. This helped explain why students had such difficulty with reconciling count and measure. I began to wonder about the relationship between counts, which are discrete, and distance, which (in this context) is continuous. Perhaps that is a paradox that children are grappling with—how can a discrete unit measure a continuous space?

Students' Thinking about Partition

I identified two students' papers that illustrated contrasting thoughts about how to represent measurement on a ruler and created a case study for the students to examine. I challenged the students to examine the two rulers and decide which one could be used to get an accurate measure of the magic marker they had on their desk. The students, in their small groups, were to test out the two rulers and determine which they would use and why (See Figure 6).

A

B

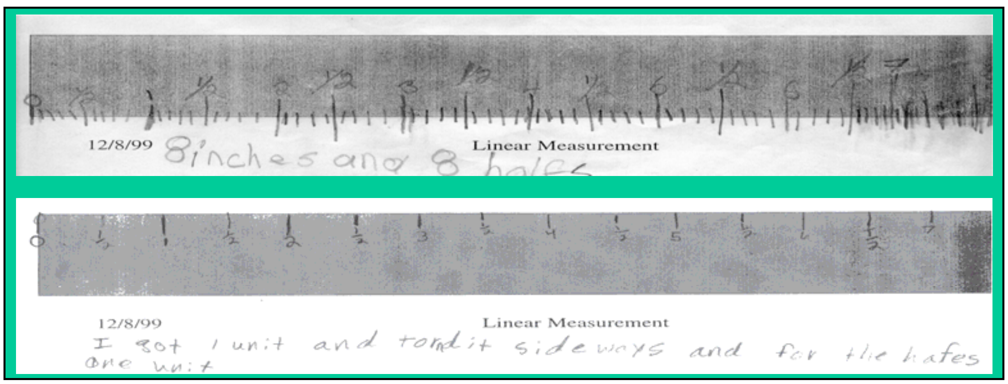


Figure 6. Alternative Ways of Representing Measure

When Rosaura looked at the two rulers, she immediately decided that ruler B would give a more accurate measurement. She concluded that the partitions have to be equal or you can't accurately determine the measure. This was a major shift in her conceptions of measure: Everything she had done up to now, including the ruler she had created on this assessment, was centered around the idea that the partitions had no meaning and that drawing in freehand lines as student A had done was not problematic. Most of the class argued that a measurement would only be accurate if the partitions were equal. One said: "With A, you don't know how big the gaps are, and if you were measuring something smaller than a unit, you wouldn't know what to call it." Another student said: "If I fold Ruler A the on the dotted line, the halves would not be equal. Even though both sides have the same number of partitions, it is not half. The lines on the ruler are there to help us to measure. You need to know where the lines go and have equal space between them."

Reflecting on My Inquiry

I made a concerted effort to set aside some time to carefully consider students' responses to the assessment tasks, to look at the notes that I had made, and to reflect about how instruction had gone up to this point. I found it helpful to pose some questions about my students:

1. How strong is their fractional understanding? If it were really strong, would that understanding transfer to linear measurement or help facilitate a stronger understanding of partitions?
2. How are ideas about fractions in equal sharing contexts the same or different than those evoked by linear measurement?
3. What experiences do the students need to have to reconcile discrete counts and continuous dimension?

After creating this list of questions and concerns, I began to think about what was important, where I needed to go next and how to get there. The big ideas I was focusing on were at a much higher level of complexity than when I started. I decided to video tape some of the class discussions, so that I could go back and get a more in-depth analysis of what happened and see how learning grew from discussion. This seemed to be where the bulk of the learning took place, as students examined and challenged each other's thinking. I decided to first focus on students' fractional understanding in different contexts.

Exploring Fractions

I wanted to know if the students' difficulty with partitioning had to do with their depth of fractional understanding. First I posed equal sharing problems, such as:

> There are 4 children at a party. There were 3 pizzas. If each child received an equal amount of pizza, how much would they each get?

All students were able to label fractional parts ranging from halves to sixteenths. They were even able to add the fractional parts and determine equality of fractions with uncommon denominators. This suggested that fractional knowledge of part-whole relationships was not the root cause of the difficulty students were having with partitions in the context of measure.

I then asked students to identify as many numbers as they could that were between 0 and 1. They were only able to generate unit fractions on the number line. So I decided to focus only on fourths to see what would happen. I asked them what fourths are between 0 and 1. Again, they placed only unit fractions (1/4 repeatedly) along the number line as in the upper portion of Figure 7.

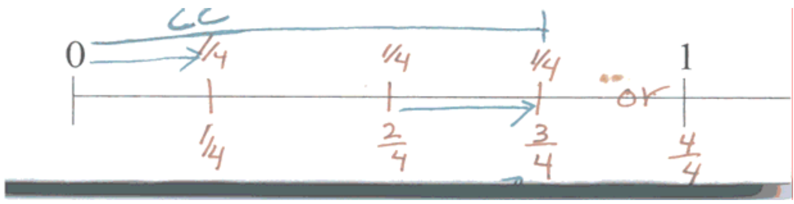


Figure 7. Initial Solutions to a Number line Problem

I questioned the class about how they got ¼ after the ½ mark. Cesar stated that he was thinking that the distance from ½ to the next line was ¼. He was not thinking about the distance from zero but about the space between the partitions (each being ¼ of a unit apart). I asked them to look at the distance from the zero; "Was the first ¼, the same distance from 0 as the third ¼ fourth. CeCe argued that it was ¾ away from the zero not ¼ away. The number line experience helped me make more sense of the incongruity I had identified previously, where students could measure and but not create a measurement tool. The students were conceiving of fractions as part-whole relationships of concrete objects, instead of thinking about ratios of distances from the zero point measured with respect to a unit. Once again, I was surprised by what students did. A few ordered fractions as: 1/2, 1/3, 1/5, 1/6, 1/7, 1/8, 1/9 and 1/10. When I asked some follow up questions about distance from 0, they seemed to think that 1/10 would be closer to 1 than would 1/2. This prompted me to consider ways of revisiting units and ratios of units in subsequent lessons.

Creating a Measuring Tape with Large Units

I again approached measure by having students design another tool, but this time I used a common unit about a foot long. I had them use this comparatively large unit, because I wanted them to continue to focus on partitioning and labeling the partitions. I also wanted to revisit the notion of a fractional distance in a context where distance was in play, rather than understood metaphorically as it is with the number line.

Putting a strain on identical units, composite units, and iteration. Once again, students measured different objects in the classroom. All students constructed tapes with identical units and congruent partitions, which suggested a major shift in the students' understanding. Follow up questioning convinced me that students understood the functions of both. So, I decided to up the ante and put a "strain" on thinking: What if we measured objects with tape-lengths? (The tape measure was a composite measure of 5 units.) This forced the students to grapple with both iteration and equal units when measuring using both units and "units of units." (This emerging emphasis on composite units later helped me re-cast multiplication as iteration of these composite units.)
One group struggled with this added layer of complexity and how to accurately communicate about their measurements. When that group measured the chalkboard, they came out with two different measurements. The first was 2 11/16 (Figure 8). They counted how many tape measures (unit of units) the board was long (2 tape measures), but then measured the remaining space using an individual unit (11/16). Maria immediately challenged the group's thinking.

"Why did you put 2 11/16 instead of 10 11/16?"

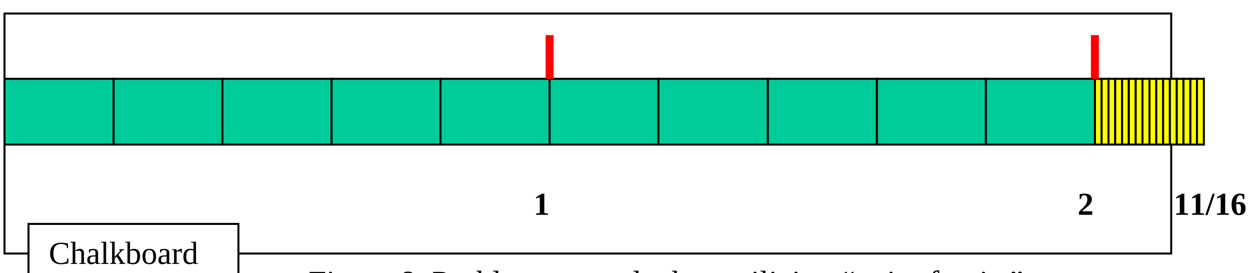


Figure 8. Problem created when utilizing "unit of units"

The class was able to respond easily to Maria's question. They all agreed that the 2 represented the number of tape measure lengths used and not the number of units. Cesar quickly added,

> *"They are mixing units. People might think that it means two units and not two tapes."*

When the same group shared their second measurement for the board, the concept of "units of units" pushed the class's thinking further and caused some students' thinking to waver as they made sense of what this group did. I tried to capitalize on this opportunity by highlighting a fraction as a ratio, where the same value (1/8) might in fact represent very different distances. The vignette below shows how “unit of units” challenged the students’ thinking about equal units, iteration, and their system of representation (See Figure 9).

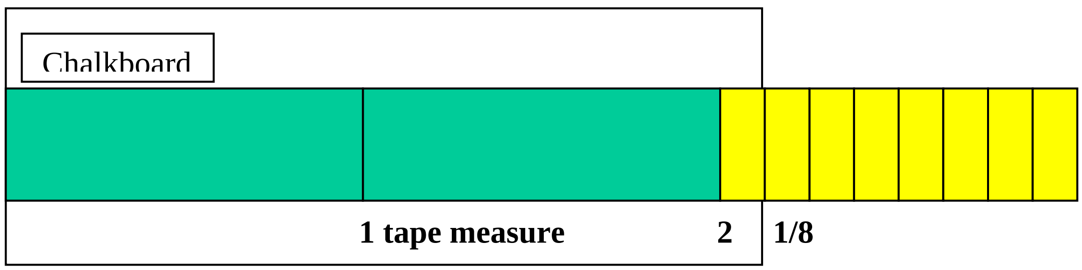


Figure 9. Revisiting Composite Units

T: So Jamie, what do you think? Is this correct? Is it acceptable to say that the board is 2 1/8 measuring tapes long?

Jamie: It doesn’t matter. You could change it from 2 1/8 to 10 1/8 because 2 tapes are actually 10 units. [Figure 10 illustrates the claim made by Jamie]

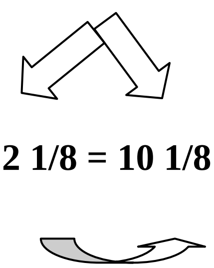


Figure 10. Jamie claims equivalence between 2 quantities of different measure

T: So you are saying that the 2 (measuring tapes) here is the same as the 10 (units) here (drawing an arrow from the 2 to the 10). Is that correct?

Jamie: Yes

T: Is this eighth the same as this eighth (pointing to the fractional part on each side of the equal sign). *No response* So Jamie is saying that this 2 is the same as this 10 because the tape measure is equal to 5 units and the board is two measuring tapes long. But my question is, “Are both 1/8’s the same or are they different?”

Cesar: They are different, because the first 1/8 will be bigger than that other 1/8. The first one is 1/8 of the tape measure and the second one is 1/8 of the unit...one is smaller than the other is so they will not be the same size.

Cesar used the yellow individual unit to determine what 1/8 of a tape measure (unit of units) would be in unit measure. He laid the unit on top of 1/8 of the measuring tape and starts counting. He determined that 1/8 of a measuring tape unit was the same as 11/16 of the original unit. This experience gave the other students in the class an opportunity to stretch their own thinking too and make further sense of the use of equal units, iteration and the importance of effective communication of measurement.

Using measure to make sense of fractions. This task not only allowed for the students to make sense of "unit of units," but it also helped the students to gain a deeper understanding of equivalent fractions, adding fractions with like denominators, and operating on fractions and ratios. For example, Adriana's group partitioned their units into 8ths and when confronted with a measure that required the use of smaller partitions, they folded their unit again (into two congruent parts) and created 16ths. However, when they measured the door, they needed to create a smaller partition of a unit to get an accurate measure. Adrianna marked where the door ended on her unit but did not know what to call it. The group taped her measuring tool on the front board so that the class could see the dilemma (Figure 11).

Adriana: (Comes up to the front of the room and points to the line on their measuring tape.) It's like this (holds the yellow unit to use to measure the incomplete unit from their door measurement).

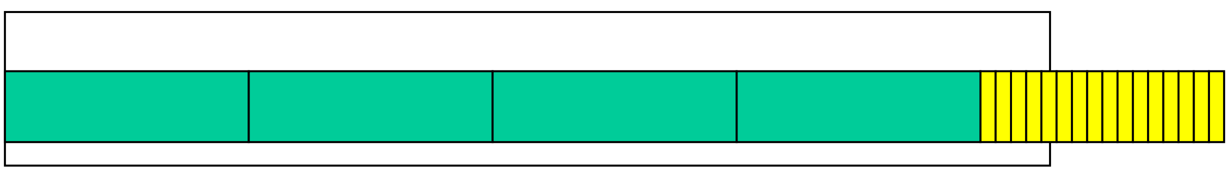


Figure 11. Adriana's Naming Dilemma

T: So how many sixteenths is this (they count together). Oh, so your line falls between the 4/16 and 5/16. So it's not a full 1/16, what can you call that? Is that what you are struggling with?

Adriana: Ah-huh

T: So how many units are we talking about?

Adriana: (She looks back at her measuring tape and counts.) Four full units...

T: So it's four full units and somewhere between 4/16 and 5/16. It's half way in between 4/16 and 5/16 and Adriana wants to know what she could call that?

Maria: 4/16 and 1/2 of 1/16 units

T: Okay, so 4 4/16 + 1/2 of 1/16 units. (Looking at Adriana) So we have one, two, three, four sixteenths and half of a sixteenth...what is that piece called?

T: What if we have 16ths but we need to go further?

Cesar: 4 4/16 plus...(short pause) 1/32 units

T: Where does that 32nds come from?

Cesar: If you fold your units in half one more time, it will be 32nds.

T: Why would I end up with 32nds?
Cesar: Because 16 + 16 are 32...
T: So what am I doing when I fold it? I am doubling my 16ths right? So is there something else that I can call this instead of 4 4/16 + 1/32 units?
Richard: 5/32
T: Is 4/16 the same as 4/32?
Cesar: 9/32
T: You think it is 9/32? So you think it should be 4 9/32 units? Where did you get the 9?
Adriana: Because 2/32=1/16
Cesar: You have 4/16 and if you times it by 2, then you end up with 8/32.
T: So you are saying that 4/16 = 8/32 plus if you add 1/32 then you end up with 9/32?

This vignette illustrates how the students began to make sense of fractions and of operations on fractions in this context. I had not taught these operations explicitly, but this episode revealed that children could:

- Figure out an equivalent fraction for 4/16
- Determine that ½ of 1/16 = 1/32
- Calculate 4/16 + 1/32 = 9/32

Equivalence. What happened next was totally unexpected. The other students in the class got very excited about figuring out other ways to describe equivalent fractions. Without any prompting, Cesar shared with the class another equivalency for 4/16. Student appeared to enjoy developing knowledge:

Cesar: 4/16 is equal to 16/64
T: Because if I cut each 32nd in half, I get 64ths, right?
Class: Yeah
Cesar: So 1/16 = 2/64
T: Are you sure about that? (Silence) 1/16 is equal to how many 32nd?
Class: 2/32
T So how can 1/16 be equal to 2/64? (Silence) So how many 32nds are in 1/16?
Class: Two
T: So it would be how many 64ths?
Class: 4
Cesar: You could do 128ths!.You would have 32/128.

As the class was going to lunch, they continued to play with the idea that you could repeatedly half the fractions and create smaller fractional pieces that were equivalent to 4/16. By the time they left for lunch, Cesar had calculated 256ths and

512ths. He was not alone. Others chipped in, revisiting some of the territory we had covered in the class but making it their own. This was icing on the cake. Although we had not yet instructed children about equivalency, Cesar and some of his classmates began to see equivalency here as a matter of repeated multiplication of both the numerator and the denominator by 2. Later expanded on this sense of building equivalent fractions by multiplying by any expression of 1 that we liked.

Giant Units

I thought if I gave the students more opportunities to work more closely on measurements that focused on partitions and zero point, it would help build their understanding. I developed yet another ruler construction task but this time, each unit was six feet long. This large standard unit allowed students to compare and contrast solutions that were easily visible. Furthermore, because I asked students to measure the heights of 8 paper cut-outs of children all less that 6 feet tall, all solutions involved partial units. After the students completed the task, they recorded their data on a class chart, so that we combined measurement and ways of displaying data. Students examined the data collected (the measures of each of the eight cut-outs, displayed in Figure 11, with my measurements in the first column) and developed questions about the data for classroom discussion. With each question posed, a new layer of understanding was revealed to me--which allowed me to challenge the students' thinking.

	Group 1	Group 2	Group 3	Group 5	Group 6	Group 7	Group 8
Child A – 15/16 giants units	8/8 + 1/2	31/32	6/8 +1/2 of 1/8	1/16, 30/32	15/16	15/16	30/32
Child B – ¼ giants	2/8	8/32	1/4	3/4, 2/8	1/4, 4/16	2/8 4/16	8/32
Child C – 5/8 giants	5/8	20/32	5/8	20/32	5/8, 11/16	5/8, 10/16	20/32
Child D – 15/32 giants	15/32	15/32	3/8 + 3/4 of 1/8	15/32	15/32	15/32	15/32
Child E – 7/16 giants	14/32	15/32	3/8 + 1/2 1/8	18/65	7/16, 14/32	11/16	14/32
Child F – 7/8 giants	7/8	28/32	7/8	1/8, 2/16 28/32	7/8, 14/16	7/8 14/16	28/32
Child G –¾ giants	6/8	24/32	2/4	6/8 , 6/4	3/4, 24/32	6/8, 12/16	3/4
Child H – ½ giants	4/8	17/32	1/2	1/2	1/2,	1/2, 4/8 2/4	1/2 8/16

Figure 11. Class Data Display of Measures with Giant Units

Although some of the differences in measure among groups were small, they were occasions for conversations that revealed much about how my students were thinking about zero-point. I learned that challenging even the slightest discrepancy in data made the students' thinking transparent and revealed a tremendous amount of information. What many of these conversations revealed was a continuing difficulty with symbolizing measure as a distance rather than as a count of units. For example, the second group measured the height of H as 17/32, a small departure from 1/2, so seemingly insignificant. However, as Figure 12 suggests, Jessica and some of her classmates again oscillated about how to label the mid-point of their 32 partitions of the giant unit.

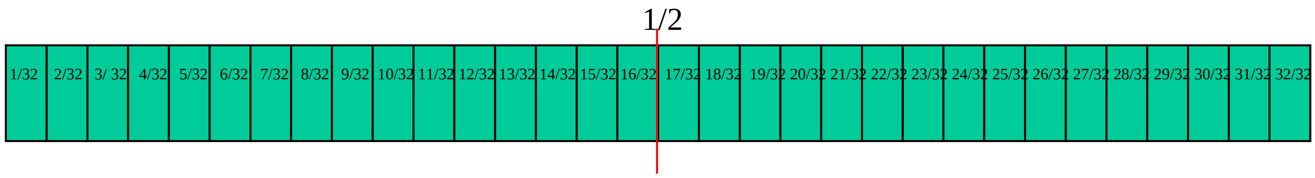


Figure 12. Symbolizing 1/2: 16/32 or 17/32?

T: What did you get?

Jessica: 17/32…it's ½.

T: How many 32ths is that?

Jessica: (Conversing with Nayla) 17/32?

T: Are you sure about that? Could you come up here and show us how you figured this out? Can you show us what you are thinking?

Jessica: (Jessica is trying to figure out how many 32ths are equal to a half by folding her tape in half repeatedly. Then she counts the boxes.)

T: So you are going to fold it in half to figure out how many 32ths are equal to 1/2?

Jessica: 16 1/2 32ths?

Ricardo: It should be 16 because that is half of 32.

T: Jessica, what are you counting, the lines or the boxes?

Jessica: The boxes...

T: So what is the answer?

Crystal: It’s 16.

T: It’s 16? Why?

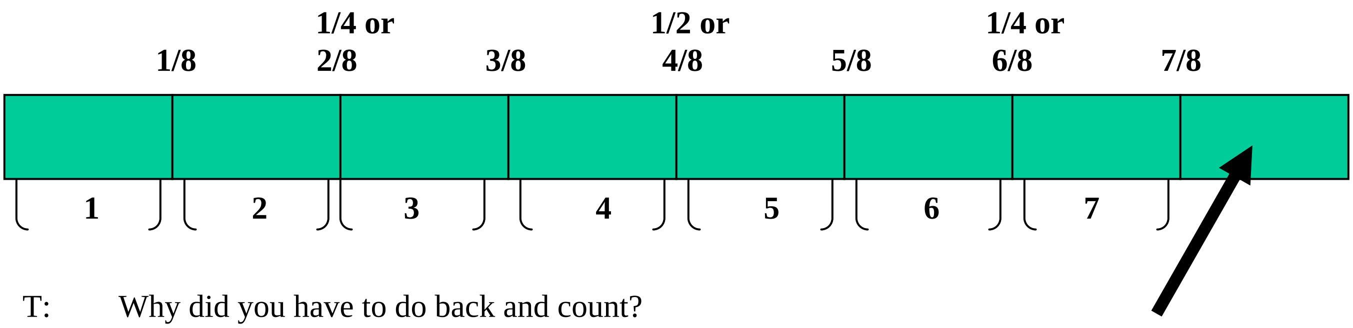
Crystal: Because there are 16/32 on one side and 16/32 on the other side.

T: So this brings up that never-ending question... Where do you number your measuring tape? Do you label the boxes or the lines? Or does it not matter? Why?

This vignette reveals some very important information about their understanding of zero point. Jessica’s group labeled the spaces because they saw each partition as a part-whole fraction, not as a distance traveled. When the measurement came between 16/32 and 17/32, Jessica’s thinking was challenged. She was not able to make sense of it at all. She was able to say the answer was ½ because if she folded the unit in half, half of the length was on each side, but seeing that 16/32 was equal to ½ was not evident because it was viewed just as a box that bordered the ½ mark. She did not realize that 16/32 was not just the 16^{th} box; it was 16/32 away from the zero point.

A variation on this challenge (of symbolizing distance) was posed by the measures generated by the third small group for child A. CeCe spoke for this group, and I was especially fascinated because CeCe had, in the contexts of whole number measures, argued forcefully for considering distance, not simply counts (expressed by placing numerals “on lines, not boxes.”) The students in this group had identified the length of

Child A as 6/8 instead of 7/8. A classmate and CeCe came up to the front of the class and measured child A for the class. CeCe looked puzzled and recounted the partitions on the giant unit. She looked confused about her answer, so she went back and began to count the partitions. Figure 13 and the vignette below illustrate how CeCe's group labeled the unit and how I attempted to shift her thinking when I saw her re-counting.



T: Why did you have to do back and count?

CeCe: Because there are 8 spaces but we only have 7/8 in lines.

T: So shouldn't you be able to come over here (pointing to the 7/8 mark on the ruler) and say that it's 7/8 without having to go all the way back and count?

CeCe: No!

T: Why not?

CeCe: Because we are marking the lines and not the spaces. Remember when we did that thing when we were talking about whether it should be the spaces that we measure or the lines that are important? This one labels the lines.

T: So when you went back to count, what did you count?

CeCe: The spaces....

T: So you went back and counted how many spaces that you used... I see. (CeCe models the space with her hands shown with the brackets, on the diagram above, holding them apart the same distance as 1/8 and went across and started counting how many spaces were used) One whole space, two, three, four, five, six, seven... Look where my hands are at the end of seven whole spaces....

CeCe: I thought that the line with 7/8 describes the next space over (indicating the space that had not counted.)

T: (Pointing to the last space on the ruler) So you thought that this space was 7/8.

CeCe: I thought...

T: So what do you think now?

CeCe: 8/8

T: So if you go all the way to the end of the ruler, what would you have?

CeCe: 8/8

Reprise. These dilemmas were surprising. I had thought that the class clearly conceived of measure as a measure of distance, not simply a count of units, and for whole numbers of units, this assumption appeared warranted. Yet these episodes suggested that this understanding had to be re-considered in light of partial units. CeCe's wavering between count and measure was especially illuminating for me. As I reflected more about this issue, I began to question how I had introduced and pursued fractions with children with only one model—part-whole. I believe that the conception of part-whole, developed mostly in fair-sharing situations, competed with the notion of a fraction as a ratio (of distance traversed as measured in particular units). In the beginning, I had questions about what this meant about children's thinking but did not actively pursue it until later. I concluded that I allowed students to repeatedly count units and partitions as discrete increments far too long without building on the idea of a continuous distance in space. (The idea of distance is more general mathematically, but that's not important for this issue). Having students symbolize their units was a window to their thinking and also a tool for promoting conceptual change. I also found that by promoting repeated splitting or partitioning of units, children often investigated equivalence.

For example, in the previous episode, CeCe's group only labeled their giant unit to eighths. When they were confronted with measurements that were smaller than 1/8, they resorted to partitions-of-partitions:

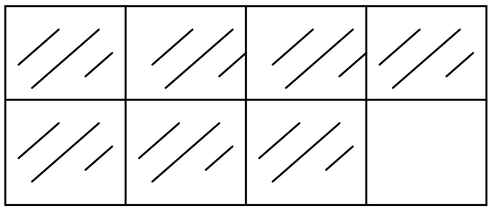
CeCe: Its 7/8 and 1/2 of 1/8.
T: So how do you know that it's half?
John: Because you can fold it (the 1/8 partition) in half.

This group repeatedly used "partitions of partitions" to describe measurements that incrementally fell between the eighths partitions. Their measurement for Child D was 3/8 and ¾ of 1/8. They divided the eighth partition folded it into four segments and the space to be measured as 3 of the 4 segments.

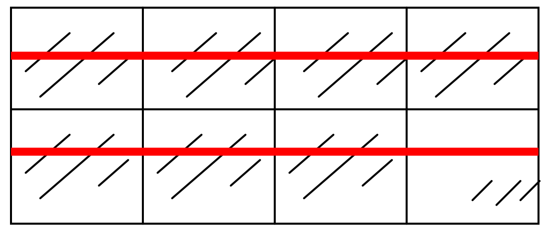
The use of "partitions of a partition" was fairly common in the class. I challenged the class to examine their thinking about "partitions of a partition" and what exactly their answers were equivalent to. I asked the class to create number sentences that would be equal to 7/8 + ½ of 1/8. The students began to find other ways to describe that amount. They explained that:

- 14/16 = 7/8
- 1/16 = ½ of 1/8
- 7/8 + ½ of 1/8 = 15/16
- 7/8 + ½ of 1/8 = 7/8 + 1/16

When the students were asked to demonstrate that their number sentences were true, they constructed diagrams and charts to illustrate their thinking. Figure 14 depicts an area representation of 7/8



= 7/8



Then they took the diagram and divided each of the 1/8's in half. And they explained that 2/16 = 1/8.

Then they took it further.

7/8 = 14/16
½ of 1/8 = 1/16
14/16 + 1/16 = 15/16

Andres showed the class that you could determine equivalence of 16ths and 1/8ths using a T-chart. He explained that every two sixteenths is equal to 1/8.

16ths	8ths
2	1
4	2
6	3
8	4
10	5
12	6
14	7
16	8

This table shows the coordination between number of partitions and the resulting equivalence. It illustrates how students' thinking about "partitions of a partition" can lead to in-depth investigation about equivalent fractions, assist them in understanding how to add fractions with common denominators and to operate on fractional numbers. The students were intuitively illustrating their understanding of multiplication of fractions. Both the diagram and the t-chart show how 1/8's can be operated on to create sixteenths (every 1/8 = 2/16).

Summing Up: Lessons Learned

I found that formative assessment was a tool that helped students develop understanding of the mathematics of linear measure even as it helped me understand enough about how they were thinking to improvise and take the next instructional step. I found too that students' responses to my tasks challenged my own understanding, both of the mathematics and of how students might be thinking and learning. Some of what I learned about the nature of learning brought home some of the conversations I had in our larger community. We talked about how learning is mediated by conceptual tools, and students' efforts to represent their actions symbolically brought this home for me. We also talked in our community about how individual learning is often characterized by variability, but our conversations took on a whole new cast when I closely examined the learning trajectories of several of my students.

The nature of measurement and its connections to other realms of mathematics were also cast in new light by my experiences in the classroom. Children needed repeated opportunities to explore the implications of the central ideas in measurement. Understandings about units needed to be re-developed for composite units. Ideas about fractions as part-whole relationships needed to be enlarged by views of fractions as measures and as ratios. When I first began working with the class, I thought that the sense of a fraction as a part-whole was sufficient, but my children taught me otherwise. I learned that conceptions of units that I had treated as fairly transparent were in fact significant intellectual achievements. I realized too that many of the problems that the students had were due to our own lack of understanding and ability to recognize issues related to the "big ideas" in measurement. I also discovered that:

- Using formative assessment effectively was much more difficult than I thought, perhaps because initially, I was driven more by implementing the tasks and what I thought they were designed for, than focusing on I could learn about what my students' thinking was telling me.
- My skills in using formative assessment were relatively underdeveloped. Using formative assessment is a skill in itself, the practice must be reflected upon, honed and continually improved upon. My first steps were marked by tentative

understandings of the implications of student thinking and by a corresponding lack of coupling between instructional planning and the formative assessments. I

- Failed to thoroughly examine all I knew about what specifically the students understood and the implications for that understanding.
- Had difficulty determining and/or understanding the significance of what was revealed in the students' thinking.
- Had difficulty determining what should be documented and how to document it.
- Failed to use some obvious information to plan instruction or to drive classroom discussions.

Students' Thinking. Although the students' thinking began to shift, the process was very slow. It did not follow a linear progression, nor was any student's journey the same as another's. On some tasks, students' strategies and reasoning were well aligned with mathematical conventions, but on others, strategies and reasoning seemed more idiosyncratic. Each of the tasks contributed to my students' development of understanding of linear measurement in different ways. They also contributed to my ability to understand what the students were thinking, what that thinking meant, and knowing what to do with that understanding.

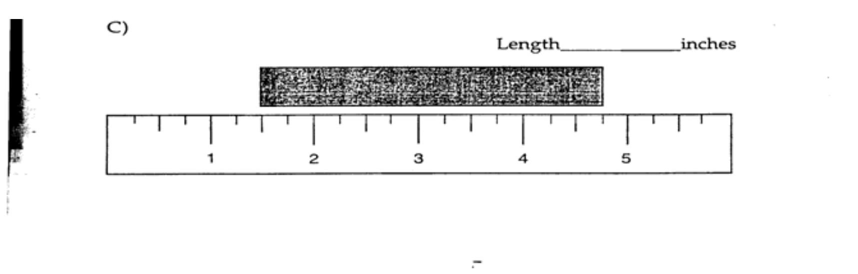
Continuity vs. discreteness. One of the biggest gaps in my understanding was how to address an apparent paradox: units and partitions are discrete but distances are continuous. One teaching strategy that seemed to help was that of repeated partitioning or splitting. The students began to develop intuitions about multiplication as splitting, rather than as repeated addition. This seemed to form a conceptual grounding for merging the discreteness of partitions with the continuity of length, because students began to see that the number of partitions could be made as large as needed for any purpose, even as the size of each partition diminished. Understanding this idea was a big step in resolving the dilemma of discretely marking what was perceptually continuous. The use of the number line, although I did not pursue it thoroughly, also seemed a good tool to further support students' thinking.

Of the big ideas in measurement, I have come to believe that zero point is the most critical. Ideas about unit, iteration of unit, and partitions of unit must be made sense of through the understanding of their relationship to an origin. I think that my focus throughout the year was for most part on congruence of units, iteration, partitions and zero point as if they were separate concepts. But when I was finally able to make sense of the students' thinking, their issues were more about the relationships of these big ideas to zero point and to each other.

When students had to grapple with composites of units, I was able to push on their thinking to help them and the rest of the class to grapple with the big ideas in ways that I had not done before. I was finally mastering the use of formative assessment by identifying and understanding the students' thinking and using that understanding to challenge their thinking.

I also found out that there are many layers of understanding in linear measurement. Making sense of linear measurement was not an all or nothing proposition. Students often understood some of the conceptual foundations of measure without yet integrating them into a coherent system. For example, Jessica appeared to understand the functions of using identical units and of creating equal-sized partitions of units for accumulating a measured quantity. However, she did not appear to integrate these understanding with the idea of an arbitrary origin.

When I completed this project, over half of the students in the class were able to create a ruler with a continuous system of representation based on zero point. Many others appeared to understand much more about the nature of unit, but fell short of my expectations. These impressions were confirmed by post-test performance on the same items. Over 60% of students could solve items A and B. The most difficult item, C, was solved by 38% of the class. Although I had hoped for more, the results indicate a significant improvement in performance. Although this post-test was originally meant to identify student growth, I once again found the power in understanding the reasoning behind the students' work. I asked the students to share with the class how they solve question C, and I was very impressed with their strategies.



Five students indicated that they could not figure out the answer with the strip in the middle of the measuring tool so they found a way to move the strip back to the zero point. This strategy told me a great deal about what the children understood about zero point and that they were taking into account both the starting and ending point of the strip.

- Three of these students moved the strip ¼ of a unit at a time keeping track of the back and the front of the strip until the strip was at the zero point. Then they were able to use the ruler to read the measure.
- Two other students traced or marked the length of the strip on another piece of paper and then laid it at the beginning of the ruler so that they could then just read the length from the ruler.

Other students used their understanding of composing and decomposing units and partitions to solve this problem.

- CeCe took off the half unit off the front of the strip and then counted the number of whole units (2). Then she saw that there were ¾ of a unit left on the back end of the strip and added ½ and ¾ units together to get 1 ¼. She added 2 + 1 ¼ = 3 ¼ units. They got 3 units and ¼ left over, 3 ¼.
- Andres, like CeCe, took the half unit off the front of the strip and counted the number of whole units. He then saw that he had a ½ unit and a ¼ unit left. He added the two halves and made a whole unit, added them all together and got 3 ¼.
- Maria along with six other students counted their units in a different way. They noted that the strip started at the ½ mark and counted units from half mark to half mark. They counted 3 whole units with ¼ of a unit left over, 3 ¼.

What was impressive was that their work revealed not only their understanding about linear measurement but their understanding of fractions as well. They not only took into account the starting and ending point but felt free to compose and decompose the units and partitions in flexible ways. I had not intended on questioning the students about the strategies that they used but a simple "how did you get your answer?" shifted this from a simple post-test revealing correct or incorrect responses to one that provided a window into the students' thinking.

More generally, I found that the tasks in linear measure supported understanding of some of the conceptual foundations of measure and number concepts as well. Many of the students gained a much more nuanced understanding of senses of fractions that went well beyond part-whole, which was evident even in the results of the post-test. At no point did I explicitly set out to teach students how to find equivalent fractions, to add, subtract or multiply fractions and yet, these operations on fractions arose spontaneously as they attempted to resolve many of the problems they encountered in the measurement tasks. In teaching, every ending is a beginning, and in this spirit, I am now attempting to create stronger links between the geometry of the number line and arithmetic, using linear measure as the intermediary.